Physics 226: Problem Set #6
Due in Class on Thursday Oct 13, 2015

1. We saw in class that when quarks and gluons fragment into jets, the soft particles created during the fragmentation process are produced in colored flux tubes. This means they have limited transverse momentum with respect to the jet axis (eg the quark or gluon direction) and that they are produced uniformly in longitudinal phase space. A consequence of this is that production is uniform in the variable called rapidity (usually written as y, although it has nothing to do with the y used in deep inelastic scattering)

$$y \equiv \frac{1}{2} \ln \left(\frac{E + p_{||}}{E - p_{||}} \right)$$

where $p_{||}$ is the particle's momentum with respect to the jet axis.

   (a) Show that a particle's rapidity is related to its velocity along the jet axis by the expression

   $$y = \text{arctanh} \left(\beta_{||} \right)$$

   where $\beta_{||}$ is the velocity (v/c in units where $c = 1$) of the particle with respect to the jet direction.

   (b) Show that the rapidity difference between two particles in a jet is invariant with respect to Lorentz boosts along the jet direction

   (c) Show that in the limit where particle masses can be neglected the rapidity y can be approximated by the expression

   $$y \approx -\ln\left(\tan\left(\theta/2\right)\right)$$

   where θ is the angle the particle makes with respect to the jet axis.

   (d) Consider $e^+e^- \rightarrow \textit{hadrons}$ in the center-of-mass frame where the energies of the initial e^+ and e^- beams are $E_{beam} = E_{cm}/2$. The distribution of particles will be approximately uniform in y between a minimum value y_{min} and a maximum value y_{max} where

$y_{min} = -y_{max}$. Using the definition of rapidity above, find an approximate value for y_{max} for hadrons of species h and mass m_h as a function of E_{beam}.

(e) Using this result, show that the average multiplicity of final state hadrons h of mass m_h is

$$n_h \propto \log\left(\frac{E_{cm}}{m_h}\right)$$

In other words, the multiplicity of hadrons grows logrithmically with the annihilation energy.

2. The fragmentation function $D_q^h(z)$ is defined as the probability that a quark q will hadronize to produce a hadron of species h with energy fraction between z and $z + dz$ of the quark's energy. These fragmentation functions must satisfy conservation of momentum and of probability so that

$$\sum_h \int_0^1 z D_q^h(z) dz = 1$$

$$\sum_h \int_{z_{min}}^1 D_q^h(z) dz = \sum_h n_h$$

where the sum is over all hadron species, z_{min} depends on the mass of the hadron and the energy of the quark ($z_{min} = m_h/E_q$) and n_h is the average number of hadrons of type h produced by the fragmentation of the quark. Fragmentation functions are often parameterized by the form

$$D_q^h(z) = \mathcal{N}\frac{(1-z)^\alpha}{z}$$

where α and $\mathcal{N}$ are constants.

(a) Show that

$$\mathcal{N} = (\alpha + 1) < z >$$

where $< z >$ is the average fraction of the quark momentum carried by hadrons of type h after fragmentation.

(b) Show that this formalism gives reproduces the result from problem 1(e):

$$n_h \propto \log\left(\frac{E_{cm}}{2m_h}\right)$$

for the process $e^+e^- \to 2$ jets.

3. One of the original papers on the discovery of the ψ [PRL **33**, 1406 (1974)] stated that the upper bound on the width of the resonance was 1.3 MeV. However, J. D. Jackson understood that every resonance is described by a Breit-Wigner. He observed that the area under the resonance curve of $\sigma(e^+e^- \to \text{hadrons})$ (Figure 1a of the ψ discovery paper) as a function of the energy $E = \sqrt{s}$ was 11.5×10^3 nb MeV, while the area under the curve $\sigma(e^+e^- \to \mu^+\mu^-)$ (Figure 1c of the same paper) was 7.2×10^2 nb MeV. Use these facts to determine the true width of the ψ and its branching ratio to $\mu^+\mu^-$. Assume branching ratio of the ψ to $\mu^+\mu^-$ is the same as the branching ratio to e^+e^- and that the only three possible decays modes of the ψ are to e^+e^- pairs, to $\mu^+\mu^-$ pairs and to hadrons.
Note # 1: Because Mark-I detector's ability to identify muons was poor, it could not unambigously tell whether the events entering Figure 1c of the ψ discovery paper were in fact muons rather than pions or kaons. Subsequent measurements have confirmed that the muon hypothesis was in fact correct. Note # 2: This problem depends on understanding the properties of a Breit-Wigner. See page 7 of lecture 13 for some hints.